Covariant Hamiltonian theory of extended MHD and applications to collisionless reconnection

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Introduction

- In astrophysical and other applications, plasma is often in a collisionless state, where resistive/viscous effects are not dominating on the time/spatial scales of interest
- XMHD is thought to be important for the formation of relativistic jets from active galactic nuclei, micro-quasars, and gamma-ray bursts [1]
- Extended MHD (XMHD) is formally 1-fluid model endowed with 2-fluid effects: electron inertia and Hall drift described by IMHD and HMHD limits respectively.
- These mutually exclusive effects were unified in a covariant Hamiltonian model [2].
- Eulerian action principle (AP) for relativistic XMHD is formulated so that constrained variations are generated by a degenerate Poisson bracket.
- For the first time, the Hamiltonian formulation of relativistic HMHD with electron thermal inertia is introduced [2]
- Relativistic HMHD (RHMHD) allows the violation of the frozen-in magnetic flux condition via an electron thermal inertia effect [2]
- Energy and helicity cascades are studied in non-relativistic 3D XMHD turbulence
- Study addresses recent interest in sub-electron scales that have become observable.

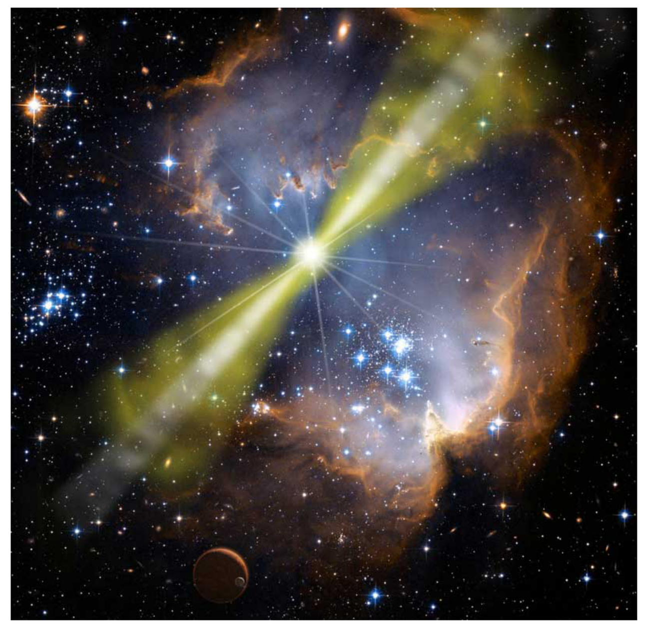

Figure: Artist's illustration of one model of the bright gamma-ray burst GRB 080319B

Advantages of Hamiltonian methods include:

- Systematic means for constructing equilibria, e.g. Beltrami flows.
- Clear derivation of reduced models avoiding introduction of spurious dissipation.
- Extraction of invariants such us helicity that plays a major role in this study.
- Understanding of how collisionless reconnection operates
- Natural means of arriving at weak turbulence theories.
- Construction of numerical integrators that automatically conserve invariants [4].

Constrained Least Action Principle

Introducing the action $S = \int \left[\frac{\mathfrak{m}^{\star\nu}\mathfrak{m}_\nu}{2nh} + \sum_{\pm} \frac{1}{2}(p_\pm - \rho_\pm) - \frac{1}{4}\mathcal{F}^{\star\mu\nu}\mathcal{F}_{\mu\nu} \right] d^4x,$ (1)

where $A^{\star\nu} := A^\nu + \frac{\Delta h}{e}u^\nu + \frac{h^\dagger}{ne^2}J^\nu$ and gen. momentum $\mathfrak{m}^{\star\nu} := nhu^\nu + (\Delta h/e)J^\nu$ with enthalpies defined as $h := h_+ + h_-$, $\Delta h := (m_-/m)h_+ - (m_+/m)h_-$, and $h^\dagger := (m_-^2/m^2)h_+ + (m_+^2/m^2)h_-$. Starting from canonical Clebsch potentials

$$m^{\star\nu} =: n\partial^\nu\phi + \sum_\pm \left(\sigma_\pm\partial^\nu\eta_\pm + \lambda_\pm\partial^\nu\varphi_\pm\right), \quad A^{\star\nu} =: \sum_\pm \left[\pm\frac{m_\mp}{men}(\sigma_\pm\partial^\nu\eta_\pm + \lambda_\pm\partial^\nu\varphi_\pm)\right]$$

with momenta and coordinates entering the Poisson bracket. The least AP ($\delta S = 0$) is equivalent to a bracket AP, i.e., $\{F[z],\, S\} = 0$ where $F[z]$ is an arbitrary functional of Clebsch potentials. We affect coordinate change to $\bar{z} = (n, \sigma_\pm, \mathfrak{m}^{\star\nu}, A^{\star\nu})$ leading to

The corresponding covariant noncanonical bracket for relativistic XMHD

$$\{F,\, G\} = -\int \left\{ n\frac{\delta G}{\delta\mathfrak{m}^{\star\nu}}\partial^\nu\frac{\delta F}{\delta n} + \sum_\pm \left[\sigma_\pm\frac{\delta G}{\delta\mathfrak{m}^{\star\nu}}\partial^\nu\frac{\delta F}{\delta\sigma_\pm} \pm \frac{m_\mp}{me}\frac{\delta F}{\delta\sigma_\pm}\frac{\delta G}{\delta A^{\star\nu}}\partial^\nu\frac{\sigma_\pm}{n}\right] + \mathfrak{m}^{\star\nu}\frac{\delta G}{\delta\mathfrak{m}^{\star\mu}}\partial^\mu\frac{\delta F}{\delta\mathfrak{m}^{\star\nu}} + \mathcal{F}^{\star\mu\nu}\frac{\delta F}{\delta\mathfrak{m}^{\star\nu}}\frac{\delta G}{\delta A^{\star\mu}} - \frac{1}{2ne}\frac{\delta F}{\delta A^{\star\mu}}\frac{\delta G}{\delta A^{\star\nu}}\mathcal{F}^{\dagger\mu\nu} - F \leftrightarrow G \right\} d^4x \quad (2)$$

here $A^{\dagger\nu} := \frac{m_+ - m_-}{m}A^\nu - \frac{h^\dagger}{e}u^\nu - \frac{\Delta h^\sharp}{ne^2}J^\nu, \quad \Delta h^\sharp := (m_-^3/m^3)h_+ - (m_+^3/m^3)h_-$ (3)

Limit to relativistic Hall MHD

Defining electron to ion mass ratio $\mu := m_e/m_i \to 0$. Applying following normalization:

$$\partial^\nu \to L^{-1}\partial^\nu, \quad n \to n_0 n, \quad T_{i,e} \to mc^2 T_{i,e}, \quad \mathcal{F}^{\mu\nu} \to \sqrt{n_0 mc^2}\mathcal{F}^{\mu\nu}, \quad c/(\omega_i L) = d_i$$

We substitute this into (2) and get HMHD bracket that generates equations:

$$\partial_\nu \left[nhu^\mu u^\nu - d_i h_e(u^\mu J^\nu + J^\mu u^\nu) + d_i^2\frac{h_e}{n}J^\mu J^\nu \right] = \partial^\mu p + J^\nu \mathcal{F}^\mu{}_\nu, \quad (4)$$

$$\left(u_\nu - d_i\frac{J_\nu}{n}\right)\mathcal{F}^{\star\mu\nu} = -d_i T_e \partial^\mu\left(\frac{\sigma_e}{n}\right), \quad \partial_\nu(\sigma_i u^\nu) = 0 = \partial_\nu\left[\sigma_e\left(u^\nu - d_i\frac{J^\nu}{n}\right)\right] \quad (5)$$

The terms including h_e must not be ignored when $h_e \gg m_e c^2$ [2].

Relativistic Collisionless Reconnection

- In IMHD, electron inertia leads to the violation of the frozen-in magnetic flux.
- This effect was suggested as a mechanism for collisionless magnetic reconnection [5]
- However, nonrelativistic HMHD does satisfy the frozen-in magnetic flux condition
- In RHMHD with $\mu \to 0$ limit electron temperature may still be large enough to allow the violation of the frozen in condition and thus reconnection.
- In [6] the relativistic e–p plasma with the assumption $\Delta h = 0$ was considered.
- In HMHD, however, [2] $\Delta h = 0$ assumption removes the aforementioned mechanism.
- The reconnection scale is expected to be $\delta \sim \sqrt{h_e d_i}$, consistent with the e-p [6]

Models	frozen-in field
(R)MHD	B
HMHD	B
IMHD	$B + \nabla\times(d_e^2 J/n)$
RHMHD	$B + \nabla\times(-d_i h_e \gamma v + d_i^2 h_e J/n)$

Table: Induction equation for nonrelativistic and relativistic (R) models.

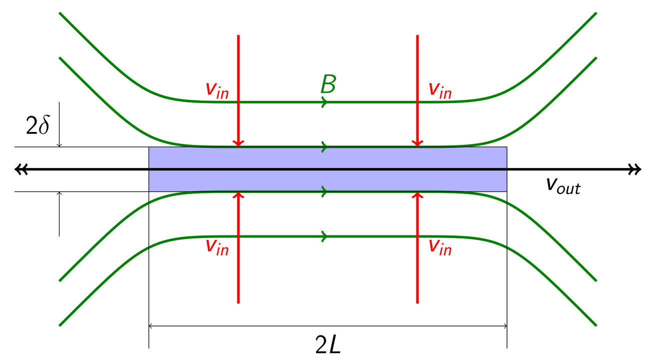

Figure: Geometry for Sweet-Parker reconnection model. Reconnection occurs in the blue layer and depends on the scale length δ.

Nonrelativistic XMHD – 3+1 decomposition

Upon rearranging the action (1) and applying relevant approximations for $v \ll c$

$$S = \int \frac{\mathfrak{m}_0\mathfrak{m}^0}{2nmc^2}d^4x - \int dx^0 \underbrace{\int \left[-\frac{\mathfrak{m}_i\mathfrak{m}^i}{2nmc^2} + n\left(\frac{1}{2}mc^2 + \mathcal{E}_+ + \mathcal{E}_-\right) - \frac{A^{\star i}J_i}{2} \right] d^3x}_{\text{Hamiltonian, } \mathscr{H}} \quad (6)$$

Performing 3+1 decomposition $F = \int \delta(x^0 - x^{0\prime})\, \mathscr{F}[n,\, \sigma_\pm,\, \mathfrak{m}^i, A^{\star i}]\, dx^0 \Rightarrow$

$$\{F,\, S\} = \frac{1}{c}\int \left(\frac{\partial\mathscr{F}}{\partial t} - \underbrace{\{\mathscr{F},\, \mathscr{H}\}^{(3)}}_{\text{3D bracket}}\right)\delta(x^0 - x^{0\prime})\, dx^0 + \int A^{\star 0}\, \underbrace{\partial^i\left(\frac{\delta\mathscr{F}}{\delta A^{\star i}}\right)}_{\text{Evaluates to zero}}\, \delta(x^0 - x^{0\prime})\, d^4x$$

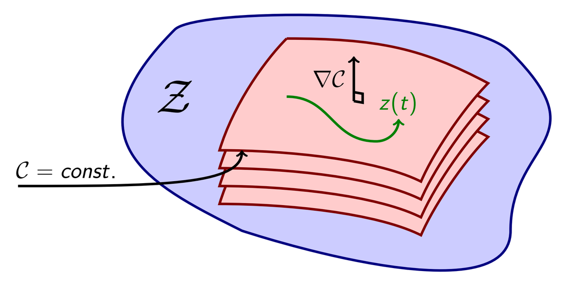

Figure: Foliation of phase space $\mathcal{Z}$ by Casimirs $\mathcal{C}$ in *finite* dimensions. Observe how dynamical system evolves ($z = z(t)$) on individual Casimir leaves. But field theories like XMHD are uncountably *infinite* dimensional!

Seeking Casimirs: $\forall\mathscr{F}:\ 0 = \{\mathscr{F}, \mathcal{C}\} \Rightarrow$

$$\mathcal{C}^{(\lambda)} = \int d^3x \left(\boldsymbol{A}^\star + \lambda\, \boldsymbol{v}\right)\cdot\left(\boldsymbol{B}^\star + \lambda\, \nabla\times\boldsymbol{v}\right),$$

where $\lambda_\pm := m_\pm c/e$. In addition when barotropic condition is violated we obtain the family (with the condition $f_{,+-} = 0$)

$$C^{(\sigma)} = \int d^3x\, \varrho\, f\left(\frac{\sigma_+}{\varrho}, \frac{\sigma_-}{\varrho}\right), \quad (7)$$

3D Turbulence in nonrelativistic extended MHD

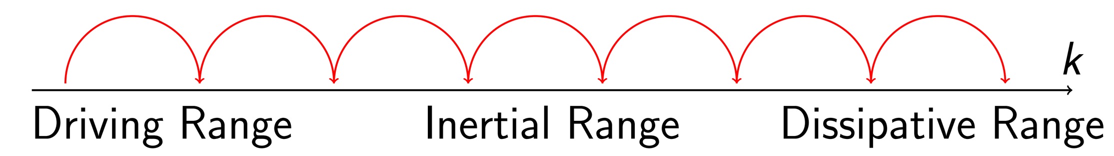

Figure: Schematics of a standard Richardson-Kolmogorov direct cascade. Energy is injected in low k, for e.g. via large scale stirring, cascades (flows) through the inertial range and dissipates at small scales (large k).

Symmetric two-point correlations (taken at $\mathbf{x}'$ and $\mathbf{x}$) of generalized helicities

$$\underbrace{0 = \frac{\partial}{\partial t}\left[\langle \boldsymbol{\mathcal{A}}'_\pm\cdot\boldsymbol{\mathcal{B}}_\pm + \boldsymbol{\mathcal{A}}_\pm\cdot\boldsymbol{\mathcal{B}}'_\pm\rangle\right]}_{\text{in a stationary regime },\mathcal{A}'_\pm := \mathcal{A}_\pm(x') = A' + \lambda_\pm v'} = -\underbrace{\langle\delta(\mathbf{V}_\pm\times\boldsymbol{\mathcal{B}}_\pm)\cdot\delta\boldsymbol{\mathcal{B}}_\pm\rangle}_{\text{helicity flux transfer rate}} + \underbrace{D}_{damping} \quad (8)$$

- More symmetry than in HMHD, where $\mathcal{C}_-$ has inverse and $\mathcal{C}_+$ has both cascades [7].
- To determine direction of cascading we investigate absolute equilibrium states [3].
- Turbulence would relax into these states if not for the continual input of energy.
- To establish a bridge between MHD [8] and XMHD results we introduce Casimirs:

magnetic- $H_M := \frac{1}{2}\frac{\kappa_+\mathcal{C}_- - \kappa_-\mathcal{C}_+}{\kappa_+ - \kappa_-} = \frac{1}{2}\int d^3x\, (\boldsymbol{A}^*\cdot\boldsymbol{B}^* + d_e^2\, \boldsymbol{V}\cdot\nabla\times\boldsymbol{V}),$ (9)

cross-helicity $H_C := \frac{1}{2}\frac{\mathcal{C}_+ - \mathcal{C}_-}{\kappa_+ - \kappa_-} = \int d^3x\, (\boldsymbol{V}\cdot\boldsymbol{B}^* + \frac{d_i}{2}\boldsymbol{V}\cdot\nabla\times\boldsymbol{V}),$ (10)

Phase space probability density $\mathscr{P} = Z^{-1}\exp[-\alpha H - \beta H_M - \gamma H_C]$ (11)

The resulting states are plotted, for HMHD set $d_e = 0$, IMHD - $d_i = 0$;

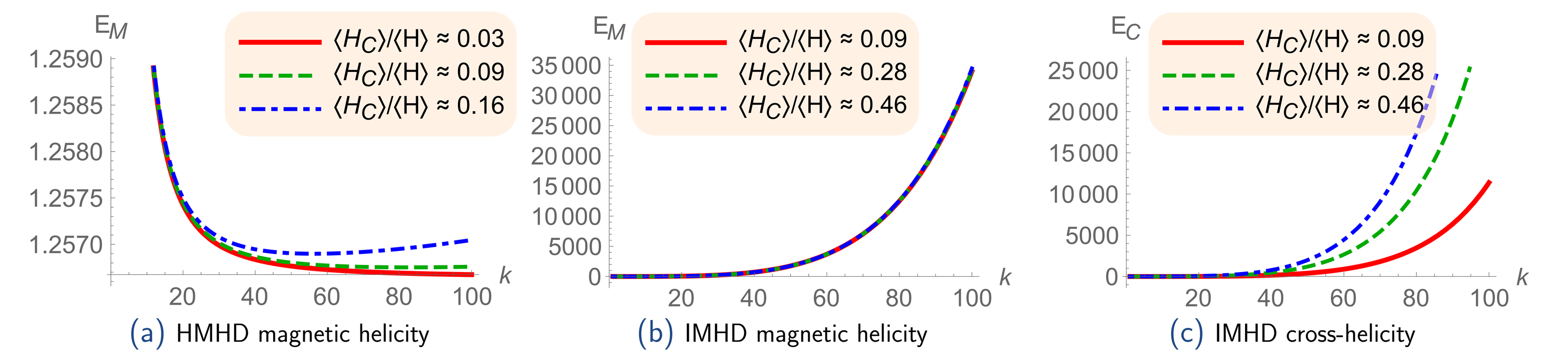

Figure: Absolute equilibria states of spectral quantities, e.g. $E_M := 4\pi k^2\langle H_M\rangle_k$ and $\langle H\rangle := \int E\, dk$. Parameters $\alpha = 10$, $\beta = 5$ and γ is varied. (a) The HMHD regime, $d_i = 0.1$. (b) IMHD magnetic helicity. (c) IMHD cross-helicity with $d_e = 0.1$.

- Inverse cascade is predicted only for magnetic helicity in HMHD range if $H_C \ll H$.
- IMHD range is characterized by direct cascades for energy, and both helicities.
- IMHD model differs from [9] where direct cascades were obtained for electron MHD.

Short Summary

We described APs for relativistic XMHD using noncanonical covariant Poisson bracket. In addition we formulated relativistic HMHD, obtained by taking a limit of the AP for XMHD. We observed that while nonrelativistic HMHD does not have a direct mechanism for collisionless reconnection, relativistic HMHD does allow the violation of the frozen-in magnetic flux condition via the electron thermal inertia effect. We estimated the resulting reconnection scale. Next we studied some general properties of non-relativistic XMHD turbulence. We expect that the (generalized) magnetic helicity undergoes inverse cascade up to a certain length scale (for a given choice of the free parameters), and then undergoes a cascade reversal. When electron inertia effects are taken to be dominant over the Hall term (IMHD regime operating on shorter sub-electron scales) we find that energy equipartition that was lost in HMHD is recovered.

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