Lorentz Transformations (Special Relativity)

A 50-minute introduction for undergraduates

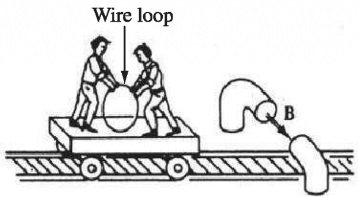
George Miloshevich

Feb 12, 2026

georgemilosh.github.com

Motivation: electromagnetism and the speed of light

- By the late 1800s, **Maxwell's equations** unified electricity and magnetism.
- A striking prediction: electromagnetic waves propagate at a **fixed speed** c.
- **Einstein's motivation (1905):** the laws of electromagnetism should look the same in every inertial frame.



Wire loop passing through a magnet

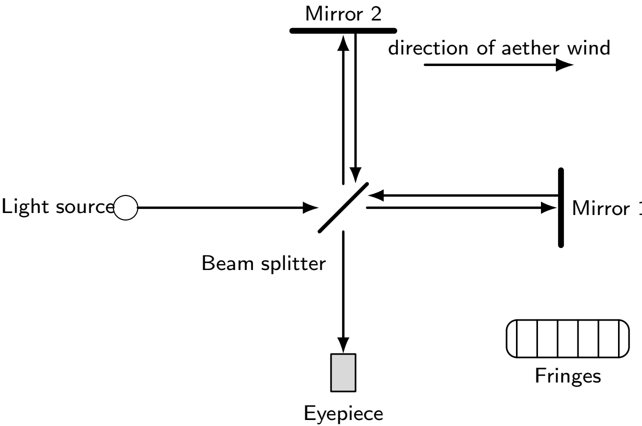
Induced emf (two viewpoints)

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

- Interpreting $\mathcal{E}$ as a circuit emf induced by a changing flux.
- Faraday's law in integral form gives the same relation.

Michelson–Morley: interferometer idea (schematic)

- Split a light beam into two perpendicular arms of equal length.
- Recombine and look for a phase shift as the apparatus rotates.
- Expectation (aether picture): an "aether wind" would change travel times.
- Observation (1887): *null result* (no detectable shift).



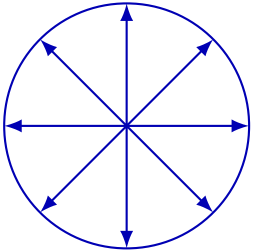
Einstein's two postulates (1905)

1. **Relativity principle:** the laws of physics are the same in all inertial frames.
2. **Constancy of c:** light in vacuum travels at speed c in all inertial frames, regardless of source motion.

Goal: find the coordinate change $(t, x) \mapsto (t', x')$ consistent with these.

(We focus on 1D motion; $y' = y$ and $z' = z$.)

Same speed in every direction



$|\mathbf{c}| = c$
in all directions

Why Galilean transformations fail for light

Assume a light pulse moves along $+x$.

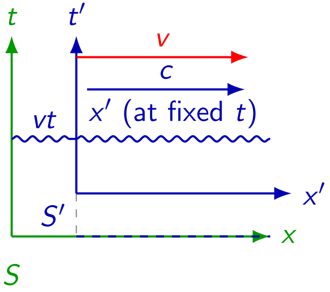
In S

$$x = ct. \tag{1}$$

Apply Galilean: $x' = x - vt$, $t' = t$

$$x' = (c - v)t = (c - v)t'. \tag{2}$$

So the speed would be $c - v$, not c.



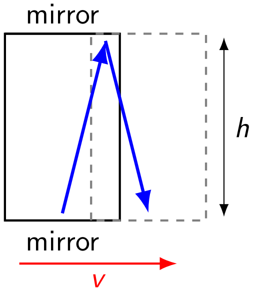
Conclusion: if c is invariant, time must transform too.

Time dilation from the postulates (light clock)

- In S' (clock's rest frame), light bounces between two mirrors separated by height h.
- One "tick" is a round trip, so $\Delta t_0 = 2h/c$.

In S, the clock moves at speed v, but light at speed c.

$$\left(\frac{c\,\Delta t}{2}\right)^2 = h^2 + \left(\frac{v\,\Delta t}{2}\right)^2 . \tag{3}$$

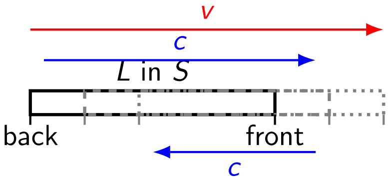


Time dilation (final result)

$$\boxed{\Delta t = \gamma\,\Delta t_0 = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}} \qquad \left(\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\right) \tag{4}$$

Let a rod be at rest in S' with proper length L_0. Put a mirror at each end.

- In S' light makes a round trip in time $\Delta t_0 = 2L_0/c$.
- In S, the rod moves at speed v and has length L.



A light pulse has travel times

$$t_{\rightarrow} = \frac{L}{c - v}, \qquad t_{\leftarrow} = \frac{L}{c + v}, \quad \Delta t \ = t_{\rightarrow} + t_{\leftarrow} = \frac{2Lc}{c^2 - v^2}. \tag{5}$$

Using time dilation ($\Delta t = \gamma \Delta t_0$) gives the final result:

Length contraction (final result)

$$\boxed{L = \frac{L_0}{\gamma} = L_0\sqrt{1 - v^2/c^2}} \qquad \left(\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\right) \tag{6}$$

Lorentz boost in x: what goes wrong classically?

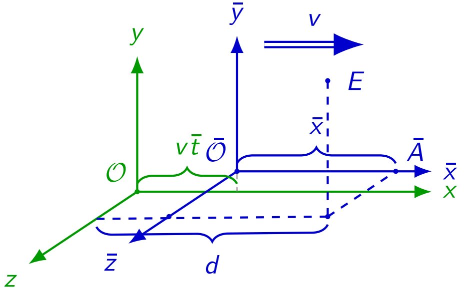
- Same segment, different frame $\Rightarrow$ a length contraction factor.

$$d = \frac{1}{\gamma}\bar{x} \Rightarrow \quad \bar{x} = \gamma(x - vt) \tag{7}$$

Run the argument from the other frame (events simultaneous in $\bar{S}$):

$$\bar{x} = \bar{d} - v\bar{t}, \qquad \bar{d} = \frac{1}{\gamma}x \tag{8}$$

$$\Rightarrow \quad x = \gamma(\bar{x} + v\bar{t}). \tag{9}$$

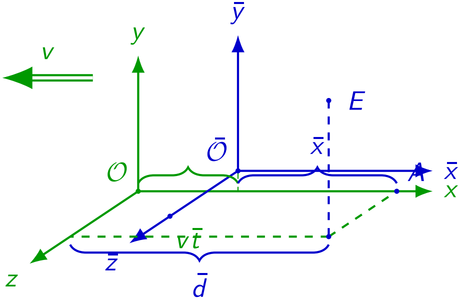


- d is the distance from $\bar{\mathcal{O}}$ to $\bar{\mathcal{A}}$ as measured in S, whereas $\bar{x}$ is same measured in $\bar{S}$.
- x is the distance from $\mathcal{O}$ to $\mathcal{A}$ in S, whereas $\bar{d}$ is the same in $\bar{S}$

Lorentz boost in x: the dictionary

Lorentz boost in x (dictionary)

Forward ($S \to \bar{S}$)	Backward ($\bar{S} \to S$)
$\bar{x} = \gamma(x - vt),$	$x = \gamma(\bar{x} + v\bar{t}),$
$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right),$	$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right),$
$\bar{y} = y, \qquad \bar{z} = z.$	$y = \bar{y}, \qquad z = \bar{z}.$



- Symmetry: swap barred/unbarred and flip $v \to -v$ to go forward $\leftrightarrow$ backward.
- The contraction input from the setup slide is $d = \bar{x}/\gamma$ and $\bar{d} = x/\gamma$ (simultaneity differs by frame).

The Lorentz factor γ and Galilean invariance

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(v/c)^2}}. \quad (10)$$

Low-speed limit ($v \ll c$): recover Galilean (x', t')

$$\gamma \approx 1 \quad \Rightarrow \quad x' = \gamma(x - vt) \approx x - vt. \quad (11) \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right) \approx t \qquad \left(\text{since } \frac{vx}{c^2} \ll t\right). \quad (12)$$

Sanity check: invariance of light

If $x = ct$, then

$$\begin{aligned} x' &= \gamma(ct - vt) = \gamma t(c - v), \\ t' &= \gamma\left(t - \frac{v}{c^2}ct\right) = \gamma t\left(1 - \frac{v}{c}\right). \end{aligned} \quad (13)$$

So $x'/t' = c$.

Relativity of simultaneity

Two events happen at different places in S but at the same time in S:

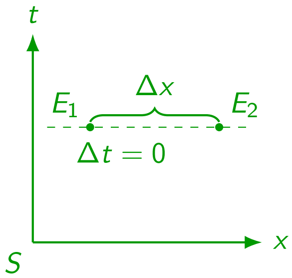
$$\Delta t = 0, \qquad \Delta x \neq 0. \tag{14}$$

From

$$t' = \gamma \left(t - \frac{v}{c^2} x \right), \tag{15}$$

we get

$$\Delta t' = \gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right) = -\gamma \frac{v}{c^2} \Delta x. \tag{16}$$



Interpretation

If $\Delta x \neq 0$ and $v \neq 0$, then $\Delta t' \neq 0$. Events simultaneous in one frame are generally *not* simultaneous in another.

Velocity addition (1D)

Suppose an object moves in S with velocity $u = \frac{\mathrm{d}x}{\mathrm{d}t}$.
Differentiate the Lorentz transformation:

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right). \tag{17}$$

Then

$$u' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - v}{1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{u - v}{1 - uv/c^2}. \tag{18}$$

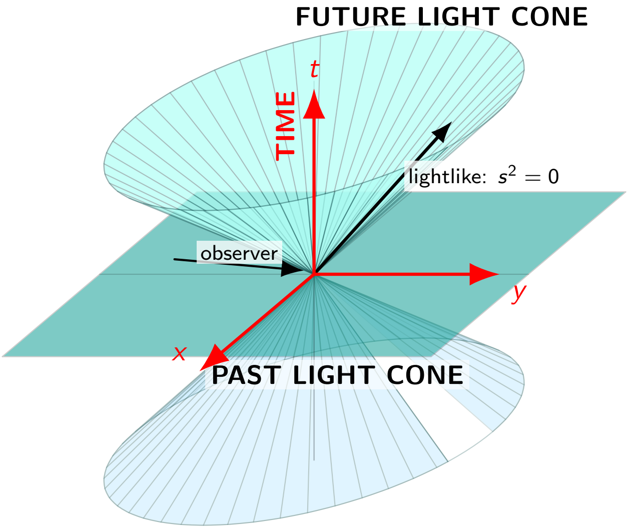
- If $u = c$, then $u' = c$.
- If u and v are less than c, so is u'.

Geometric view: Minkowski spacetime

- Combine space and time into a single spacetime.
- Define the **spacetime interval**

$$s^2 = c^2 \Delta t^2 - \Delta x^2 \quad \text{(1D)}.$$

- Lorentz transformations preserve s^2 (like rotations preserve lengths).
- Light rays have $s^2 = 0$ (the **light cone**).



Summary (and what to remember)

Standard boost (along x)

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right). \tag{19}$$

Key consequences

- time dilation: $\Delta t = \gamma \Delta t_0$,
- length contraction: $L = L_0/\gamma$,
- relativity of simultaneity: $\Delta t' = -\gamma \frac{v}{c^2}\Delta x$ (when $\Delta t = 0$),
- velocity addition: $u' = (u - v)/(1 - uv/c^2)$.

Geometry viewpoint

Preserve the spacetime interval: $s^2 = c^2\Delta t^2 - \Delta x^2$.

Source: Griffiths, *Introduction to Electrodynamics*